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**Trees with paired-domination number
twice their domination number**

by

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Trees with paired-domination number twice their domination number

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Abstract

In this paper, we continue the study of paired-domination in graphs introduced by Haynes and Slater (Networks 32 (1998) 199–206). A paired-dominating set of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G is the minimum cardinality of a paired-dominating set of G . For $k \geq 2$, a k -packing in G is a set S of vertices of G that are pairwise at distance greater than k apart. The k -packing number of G is the maximum cardinality of a k -packing in G . Haynes and Slater observed that the paired-domination number is bounded above by twice the domination number. We give a constructive characterization of the trees attaining this bound that uses labelings of the vertices. The key to our characterization is the observation that the trees with paired-domination number twice their domination number are precisely the trees with 2-packing number equal to their 3-packing number.

Keywords: domination, packing number, paired-domination

AMS subject classification: 05C69

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1 Introduction

In this paper, we continue the study of domination and paired-domination in graphs. For a graph $G = (V, E)$, a set S is a *dominating set*, denoted DS, if every vertex in $V \setminus S$ has a neighbor in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G . We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$ -*set*. Domination and its many variations have been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [8, 9]. We are interested in a variation of domination called paired-domination.

A *matching* in a graph G is a set of independent edges in G . A *perfect matching* M in G is a matching in G such that every vertex of G is incident to a vertex of M . A *paired-dominating set*, denoted PDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired* in S . Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The *paired-domination number* of G , denoted by $\gamma_{\text{pr}}(G)$, is the minimum cardinality of a PDS. A PDS of cardinality $\gamma_{\text{pr}}(G)$ we call a $\gamma_{\text{pr}}(G)$ -set. Paired-domination was introduced by Haynes and Slater in [10, 11] as a model for assigning backups to guards for security purposes, and is studied, for example, in [2, 3, 4, 6, 7, 13, 15, 16] and elsewhere.

In general we follow the notation and graph theory terminology in [8]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E . For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. If $v \in S$, then a vertex $w \in V \setminus S$ is an *external private neighbor* of v (with respect to S) if $N(w) \cap S = \{v\}$; and the *external private neighbor set of v with respect to S* , denoted $\text{epn}(v, S)$, is the set of all external private neighbors of v . We denote the degree of v in G by $d_G(v)$. We denote a path on n vertices by P_n .

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T , we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v , and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v . A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *strong support vertex* is adjacent to at least two leaves.

For $k \geq 2$, Meir and Moon [14] defined a *k-packing* in a graph G as a set S of vertices of G that are pairwise at distance greater than k apart, i.e., if $u, v \in S$, then $d_G(u, v) > k$. A 1-packing is an independent set. The *k-packing number* of G , denote $\rho_k(G)$, is the maximum cardinality of a k -packing in G . We call a k -packing of cardinality $\rho_k(G)$ a $\rho_k(G)$ -*set*. By definition, we have the following observation.

Observation 1 For $k \geq 1$ and for any graph G , $\rho_k(G) \geq \rho_{k+1}(G)$.

A 2-packing is also called a *packing* and the 2-packing number is also called the *packing number* denoted by $\rho(G)$. Thus a $\rho(G)$ -set is a packing of maximum cardinality $\rho(G)$. By Observation 1, $\rho(G) \geq \rho_3(G)$.

Definition 1 We define a graph G to be a $(\rho = \rho_3)$ -graph if $\rho(G) = \rho_3(G)$.

The following bound was first observed by Haynes and Slater [10]. The proof follows from the fact that every graph without isolated vertices has a minimum DS in which each vertex has an external private neighbor.

Observation 2 ([10]) For every graph G without isolated vertices, $\gamma_{\text{pr}}(G) \leq 2\gamma(G)$.

We call the graphs obtaining the upper bound in Observation 2, $(\gamma_{\text{pr}} = 2\gamma)$ -graphs. It is our goal in this paper to characterize $(\gamma_{\text{pr}} = 2\gamma)$ -trees. Recently, a characterization of $(\gamma_{\text{pr}} = 2\gamma)$ -trees was given in [12]. The characterization we present here is a constructive characterization using labelings that is simpler than that presented in [12]. The key to our characterization is the observation that the family of $(\gamma_{\text{pr}} = 2\gamma)$ -trees is precisely the family of $(\rho = \rho_3)$ -trees.

Observation 3 Let T be a tree of order at least 2. Then, T is a $(\gamma_{\text{pr}} = 2\gamma)$ -tree if and only if T is a $(\rho = \rho_3)$ -tree.

Observation 3 is an immediate consequence of the following two results.¹

Theorem 1 (Moon and Meir [14]) For every tree T , $\gamma(T) = \rho(T)$.

Theorem 2 (Bresar et. al. [1]) For every tree T of order at least 2, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$.

Hence, by Observation 3, to characterize the $(\gamma_{\text{pr}} = 2\gamma)$ -trees it suffices to characterize the $(\rho = \rho_3)$ -trees.

2 $(\rho = \rho_3)$ -Trees

Our aim in this section is to give a constructive characterization of $(\rho = \rho_3)$ -trees (and hence, by Observation 3, of $(\gamma_{\text{pr}} = 2\gamma)$ -trees). The main idea to our constructive characterization is to find a labeling of the vertices that indicates the roles each vertex plays in the sets associated with both parameters. This idea of labeling the vertices is introduced in [5], where trees with equal domination and independent domination numbers as well as trees with equal domination and total domination numbers are characterized.

¹Since the result of Bresar et. al. [1] may not be readily available to the referees, we provide a proof—for refereeing purposes only—of Theorem 2 in the appendix.

2.1 Main Result

By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. We define a *labeling* of a tree T as a weak partition $S = (S_A, S_B, S_C, S_D)$ of $V(T)$. The *label* or *status* of a vertex v , denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. Our aim is to describe a procedure to build $(\rho = \rho_3)$ -trees in terms of labelings.

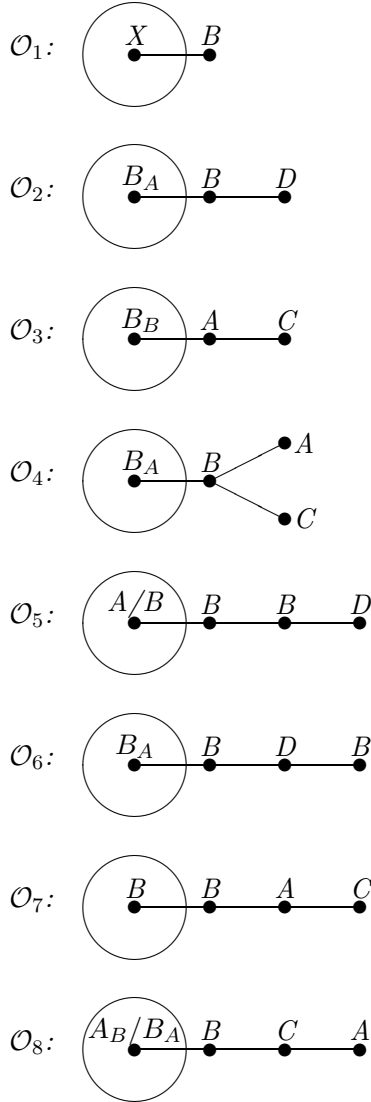
We begin by defining three labeled trees T_1 , T_2 and T_3 as follows. Let T_1 be a P_2 with one vertex labeled A and the other vertex C . Let T_2 be a P_2 with one vertex labeled B and the other vertex D . Let T_3 be a P_3 with the central vertex labeled B and with one leaf labeled A and the other leaf C .

Definition 2 Let \mathcal{T} be the family of labeled trees that: (i) contains the labeled trees T_1 , T_2 and T_3 ; and (ii) is closed under the eleven operations \mathcal{O}_j ($j = 1, 2, \dots, 11$) listed below, which extend the tree T by attaching a tree to the vertex $v \in V(T)$.

- **Operation \mathcal{O}_1 .** Assume v is a support vertex of T . Add a new vertex u of status B and the edge uv . (Note that v can have any status.)
- **Operation \mathcal{O}_2 .** Assume v and all its neighbors have status B , except possibly for one neighbor of v which has status A . Add a path u_1, u_2 and the edge vu_1 . Let $\text{sta}(u_1) = B$ and $\text{sta}(u_2) = D$.
- **Operation \mathcal{O}_3 .** Assume v and all its neighbors have status B . Add a path u_1, u_2 and the edge vu_1 . Let $\text{sta}(u_1) = A$ and $\text{sta}(u_2) = C$.
- **Operation \mathcal{O}_4 .** Assume v and all its neighbors have status B , except possibly for one neighbor of v which has status A . Add a path u_1, u_2, u_3 and the edge vu_2 . Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = B$, and $\text{sta}(u_3) = C$.
- **Operation \mathcal{O}_5 .** Assume v has status A or B . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = \text{sta}(u_2) = B$ and $\text{sta}(u_3) = D$.
- **Operation \mathcal{O}_6 .** Assume v and all its neighbors have status B , except possibly for one neighbor of v which has status A . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = D$ and $\text{sta}(u_3) = B$.
- **Operation \mathcal{O}_7 .** Assume v has status B . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = A$ and $\text{sta}(u_3) = C$.
- **Operation \mathcal{O}_8 .** Assume v has status A and all its neighbors have status B or assume v and all its neighbors have status B , except possibly for one neighbor of v which has status A . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = B$, $\text{sta}(u_2) = C$ and $\text{sta}(u_3) = A$.
- **Operation \mathcal{O}_9 .** Assume v and all its neighbors have status B , except possibly for one neighbor of v which has status C . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = B$ and $\text{sta}(u_3) = C$.

- **Operation \mathcal{O}_{10} .** Assume v and all its neighbors have status B . Add a path u_1, u_2, u_3 and the edge vu_1 . Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = C$ and $\text{sta}(u_3) = B$.
- **Operation \mathcal{O}_{11} .** Assume v has status A or B . Add a star $K_{1,3}$ with center u and leaves u_1, u_2 and u_3 , and add the edge vu_1 . Let $\text{sta}(u) = \text{sta}(u_1) = B$, $\text{sta}(u_2) = A$, $\text{sta}(u_3) = C$.

These eleven operations are illustrated in Figure 1 where by status X we mean a support vertex of any status (namely, A , B , C or D). For $Y \in \{A, B\}$, by status Y_B we mean a vertex of status Y all of whose neighbors have status B . For $Z \in \{A, C\}$, by status B_Z we mean a vertex of status B all of whose neighbors have status B , except possibly for one neighbor which has status Z .



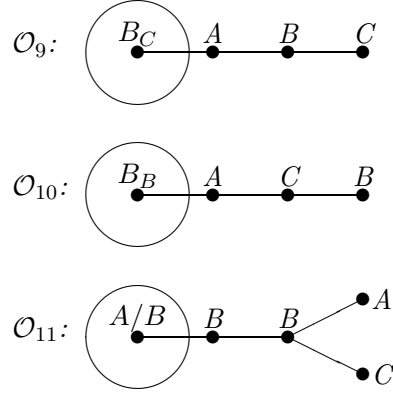


Figure 1. The eleven operations.

Our main result is the following constructive characterization of $(\rho = \rho_3)$ -trees that uses labelings.

Theorem 3 *The $(\rho = \rho_3)$ -trees are precisely those trees T such that $(T, S) \in \mathcal{T}$ for some labeling S .*

Theorem 3 is an immediate consequence of Theorem 4 presented in Section 2.2 and Theorem 5 in Section 2.3.

2.2 (ρ, ρ_3) -labelings

Definition 3 *We define a (ρ, ρ_3) -labeling of a tree $T = (V, E)$ as a weak partition $S = (S_A, S_B, S_C, S_D)$ of V such that*

- $S_A \cup S_D$ is a $\rho(T)$ -set,
- $S_C \cup S_D$ is a 3-packing, and
- $|S_A| = |S_C|$.

We are now in a position to characterize $(\rho = \rho_3)$ -trees in terms of (ρ, ρ_3) -labelings.

Theorem 4 *A tree is a $(\rho = \rho_3)$ -tree if and only if has a (ρ, ρ_3) -labeling.*

Proof. Suppose T is a tree that has a (ρ, ρ_3) -labeling (S_A, S_B, S_C, S_D) . Then, $\rho_3(T) \leq \rho(T) = |S_A \cup S_D| = |S_C \cup S_D| \leq \rho_3(T)$. Hence we must have equality throughout this inequality chain. In particular, $\rho_3(T) = \rho(T)$; that is, T is a $(\rho = \rho_3)$ -tree.

Conversely, suppose that T is a $(\rho = \rho_3)$ -tree. Let P be a $\rho(T)$ -set and let L be a $\rho_3(T)$ -set. Let $S_A = P \setminus L$, $S_B = V \setminus (P \cup L)$, $S_C = L \setminus P$, and $S_D = L \cap P$. Then, $P = S_A \cup S_D$ is a $\rho(T)$ -set and $L = S_C \cup S_D$ is a 3-packing. Further since $|P| = \rho(T) = \rho_3(T) = |L|$, we have $|S_A| = |S_C|$. Hence, (S_A, S_B, S_C, S_D) is a (ρ, ρ_3) -labeling of T . \square

2.3 (ρ, ρ_3) -labelled Trees

Given a (ρ, ρ_3) -labeling $S = (S_A, S_B, S_C, S_D)$ of a tree T , we will refer to the pair (T, S) as a (ρ, ρ_3) -labelled **tree**. In this subsection, we characterize (ρ, ρ_3) -labelled trees.

Theorem 5 *A labeled tree is a (ρ, ρ_3) -labelled tree if and only if it is in \mathcal{T} .*

Proof. It is clear that the eleven operations \mathcal{O}_i , $1 \leq i \leq 11$, preserve a (ρ, ρ_3) -labeling, whence every element of \mathcal{T} is a (ρ, ρ_3) -labelled tree.

Conversely, the proof that every (ρ, ρ_3) -labelled tree (T, S) belongs to \mathcal{T} is by induction on the order of T . The smallest (ρ, ρ_3) -labelled trees are the labeled trees T_1 and T_2 defined earlier, both of which are in \mathcal{T} . So fix a (ρ, ρ_3) -labelled tree (T, S) of order at least 3, and assume that any smaller (ρ, ρ_3) -labelled tree is in \mathcal{T} . By Theorem 4, T is a $(\rho = \rho_3)$ -tree.

To complete the proof, we need to identify a set P of vertices that can be pruned to leave a (ρ, ρ_3) -labelled tree (by induction, this pruned tree is in \mathcal{T}) and an operation \mathcal{R} or a sequence of operations that restores the pruned vertices.

If $T = P_3$, then either $T = T_3$ or T can be obtained from either T_1 or T_2 by operation \mathcal{O}_1 , implying that $T \in \mathcal{T}$.

Suppose there is a leaf u in S_B adjacent to a strong support vertex. Let $T' = T - u$ and let S' be the restriction of S to T' . Then, (T', S') is a (ρ, ρ_3) -labelled tree, and we can take $P = \{u\}$ and $\mathcal{R} = \mathcal{O}_1$. So we may assume that no leaf of status B is adjacent to a strong support vertex. Hence we may assume that $\text{diam}(T) \geq 3$.

Among all longest paths in T , let $v_0, v_1, v_2, \dots, v_{\text{diam}(T)}$ be chosen so that $\deg v_1$ is as large as possible. Let T be rooted at the vertex $v_{\text{diam}(T)}$.

Case 1. $d_T(v_1) \geq 3$, i.e., v_1 is a strong support vertex. By assumption, no leaf of v_1 has status B . Since $S_A \cup S_D$ is a packing and $S_C \cup S_D$ is a 3-packing, v_1 therefore has exactly two children, say u_0 and v_0 , one of status A and the other of status C . We may assume $\text{sta}(u_0) = A$ and $\text{sta}(v_0) = C$. Further, $\text{sta}(v_1) = \text{sta}(v_2) = B$ and $\text{sta}(w) = B$ for each $w \in N(v_2)$, except possibly for one neighbor of v_2 which has status A .

Let $T' = T - D[v_1]$. Then, $(S_A \cup S_D) \setminus \{u_0\}$ is a packing in T' , and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then (T', S') is a (ρ, ρ_3) -labelled tree where S' is the restriction of S to T' , and we can take $P = D[v_1]$ and $\mathcal{R} = \mathcal{O}_4$. Hence we may assume that $\rho(T') = \rho(T)$. Thus, by Theorem 1, $\gamma(T') = \gamma(T)$. Let D be a $\gamma(T)$ -set. Then, $v_1 \in D$. If $v_2 \notin \text{epn}(v, D)$, then $D \setminus \{v_1\}$ is a DS of T' of cardinality $\gamma(T) - 1$, contrary to assumption. Hence, for every $\gamma(T)$ -set D of T we must have that $v_2 \in \text{epn}(v, D)$. This implies that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_A \cup S_B$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_{11}$.

Case 2. $d_T(v_1) = 2$. We consider four possibilities.

Case 2.1. $v_0 \in S_D$. Then, $\{v_1, v_2\} \subseteq S_B$ and every neighbor of v_2 is in S_B , except possibly for one neighbor which is in S_A . Let $T' = T - \{v_0, v_1\}$. Then, $(S_A \cup S_D) \setminus \{v_0\}$ is a packing in T' , and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then (T', S') is a (ρ, ρ_3) -labelled tree where S' is the restriction of S to T' , and we can take $P = \{v_0, v_1\}$ and $\mathcal{R} = \mathcal{O}_2$. Hence we may assume that $\rho(T') = \rho(T)$. This implies, as argued in Case 1, that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_A \cup S_B$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_5$.

Case 2.2. $v_0 \in S_C$. Then either $v_1 \in S_A$ or $v_1 \in S_B$.

Suppose first that $v_1 \in S_A$. Then, $v_2 \in S_B$ and every neighbor of v_2 except for v_1 is in S_B . Let $T' = T - \{v_0, v_1\}$. Then, $(S_A \cup S_D) \setminus \{v_0\}$ is a packing in T' , and so $\rho(T') \geq \rho(T) - 1$. If $\rho(T') = \rho(T) - 1$, then (T', S') is a (ρ, ρ_3) -labelled tree where S' is the restriction of S to T' , and we can take $P = \{v_0, v_1\}$ and $\mathcal{R} = \mathcal{O}_3$. Hence we may assume that $\rho(T') = \rho(T)$. This implies that $\gamma(T') = \gamma(T)$ and hence that $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. As observed earlier, $v_3 \in S_B$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_7$.

Suppose secondly that $v_1 \in S_B$. If $v_2 \in S_B$, then $S_A \cup S_D \cup \{v_0\}$ is a packing in T of cardinality $\rho(T) + 1$, which is impossible. Hence, $v_2 \in S_A$. Thus, $N(v_2) \subseteq S_B$. Suppose $d_T(v_2) \geq 3$. Since $S_A \cup S_D$ is a packing, no descendant of v_2 is in $S_A \cup S_D$. Hence if u_1 is a child of v_2 different from v_1 , then $(S_A \cup S_D \cup \{u_1, v_0\}) \setminus \{v_2\}$ is a packing in T of cardinality $\rho(T) + 1$, which is impossible. Therefore, $d_T(v_2) = 2$. As observed earlier, $v_3 \in S_B$. If $T = P_4$, then $\rho(T) = 2$, contradicting the fact that $S_A \cup S_D = \{v_2\}$ is a $\rho(T)$ -set. Hence, $n \geq 5$. Every neighbor of v_3 different from v_2 has status B , except possibly for one neighbor of v_3 which has status C . We now consider the tree $T' = T - D[v_2]$. Let S' be the restriction of S to T' . Then, (T', S') is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_9$.

Case 2.3. $v_0 \in S_A$. Then either $v_1 \in S_B$ or $v_1 \in S_C$.

Suppose that $v_1 \in S_B$. If $v_2 \in S_B$, then the set $(S_C \cup S_D) \cup \{v_0\}$ is a packing in T of cardinality $|S_C| + |S_D| + 1 = |S_A| + |S_D| + 1 = \rho(T) + 1$, which is impossible. Hence, $v_2 \in S_C$. Since $S_C \cup S_D$ is a 3-packing, the vertex v_3 is therefore at distance at least 3 from every vertex in $(S_C \cup S_D) \setminus \{v_2\}$. But this implies that $(S_C \cup S_D \cup \{v_0, v_3\}) \setminus \{v_2\}$ is a packing in T of cardinality $|S_C| + |S_D| + 1 = \rho(T) + 1$, which is impossible. Hence, $v_1 \notin S_B$, implying that $v_1 \in S_C$ and $v_2 \in S_B$.

Let $T' = T - \{v_0, v_1\}$. Then, $(S_C \cup S_D \cup \{v_2\}) \setminus \{v_1\}$ is a packing in T' , and so by Theorem 1, $\gamma(T') = \rho(T') \geq |S_C \cup S_D| = |S_A \cup S_D| = \rho(T) = \gamma(T)$. Consequently, $\gamma(T') = \gamma(T)$. This implies that $d_T(v_2) = 2$. Since $S_C \cup S_D$ is a 3-packing, we observe that either v_3 has status A and all its neighbors have status B or v_3 and all its neighbors have status B , except possibly for one neighbor of v which has status A . We now consider the tree $T^* = T - D[v_2]$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_8$.

Case 2.4. $v_0 \in S_B$.

Suppose that $v_1 \in S_A \cup S_B \cup S_C$. Let $T' = T - \{v_0, v_1\}$. If $v_1 \in S_A \cup S_B$, then $S_C \cup S_D$ is a packing in T' , while if $v_1 \in S_B \cup S_C$, then $S_A \cup S_D$ is a packing in T' . It follows that $\rho(T') = \rho(T)$ and therefore that $d_T(v_2) = 2$. If $v_2 \in S_D$, then every vertex at distance 1 and 2 from v_2 is in S_B , while every vertex at distance 3 from v_2 is in $S_A \cup S_B$. But then $(S_C \cup S_D \cup \{v_0, v_3\}) \setminus \{v_2\}$ is a packing in T of cardinality $\rho(T) + 1$, which is impossible. Hence, $\{v_1, v_2\} \cap S_D = \emptyset$. If $\{v_1, v_2\} \cap S_A = \emptyset$, then $S_A \cup S_D \cup \{v_0\}$ is a packing in T of cardinality $\rho(T) + 1$, which is impossible. If $\{v_1, v_2\} \cap S_C = \emptyset$, then $S_C \cup S_D \cup \{v_0\}$ is a packing in T of cardinality $\rho(T) + 1$, which is impossible. Hence, either $v_1 \in S_A$ and $v_2 \in S_C$ or $v_1 \in S_C$ and $v_2 \in S_A$. The former case cannot occur because $v_1 \in S_A$ implies that $v_3 \in S_B$ and that every neighbor of v_3 different from v_2 is in S_B except possibly one v_3 -neighbor which may belong to S_A . But now $(S_C \cup S_D) \setminus \{v_2\} \cup \{v_0, v_3\}$ is a packing in T of cardinality $\rho(T) + 1$, a contradiction. Thus the latter case $v_1 \in S_C$, $v_2 \in S_A$ occurs and every vertex at distance 1 and 2 from v_2 except for v_1 is labelled B. Recall $d_T(v_2) = 2$. We now consider the tree $T^* = T - D[v_2]$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_{10}$.

Hence we may assume that $v_1 \in S_D$. With our earlier assumptions, we may therefore assume that every leaf of a path of length $\text{diam}(T)$ has status B and its neighbor (of degree-2) has status D . Suppose that $d_T(v_2) \geq 3$. Let u_1 be a child of v_2 different from v_1 . If u_1 is not a leaf, then, by our earlier assumptions, u_1 has status D , and so we have two vertices of status D at distance 2 apart, contradicting the fact that S_D is a packing. Hence, u_1 is a leaf and $u_1 \in S_B$. But then $(S_A \cup S_D \cup \{u_1, v_0\}) \setminus \{v_1\}$ is a packing in T of cardinality $\rho(T) + 1$, a contradiction. Hence, $d_T(v_2) = 2$. Since $v_1 \in S_D$, $\{v_2, v_3\} \subset S_B$ and every neighbor of v_3 have status B , except possibly for one neighbor of v_3 which has status A . We now consider the tree $T^* = T - D[v_2]$. Let S^* be the restriction of S to T^* . Then, (T^*, S^*) is a (ρ, ρ_3) -labelled tree, and we can take $P = D[v_2]$ and $\mathcal{R} = \mathcal{O}_6$. \square

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Appendix: For refereeing purposes only

In this appendix we present a proof of the result due to Bresar et. al. [1] that the paired domination number of a tree is equal to twice its 3-packing number. We begin with two observations from [1]. Recall that the *independence number* $\beta(G)$ of a graph G is the maximum cardinality of a set of independent vertices in G .

Observation 4 (Bresar et. al. [1]) *If D is a $\gamma_{\text{pr}}(G)$ -set in a graph G without isolated vertices, then $|D| \geq 2\beta(G[D])$.*

Proof. Let $D' \subset D$ be a maximum independent set in $G[D]$. Since each vertex of D' has a partner, and these partners are distinct, $\gamma_{\text{pr}}(G) = |D| \geq 2|D'| = 2\beta(G[D])$. \square

Observation 5 (Bresar et. al. [1]) *For any graph G without isolated vertices, $\gamma_{\text{pr}}(G) \geq 2\rho_3(G)$.*

Proof. Let D be a $\gamma_{\text{pr}}(G)$ -set and let S be a $\rho_3(G)$ -set. For each vertex $v \in S$, let v' be a vertex of D that dominates v and let $D' = \cup_{v \in S} \{v'\}$. Since the vertices in S are pairwise at distance at least 4 apart, the vertices v' , where $v \in S$, are distinct and the set D' is an independent set in $G[D]$. Hence, by Observation 4, $\gamma_{\text{pr}}(G) = |D| \geq 2|D'| = 2\rho_3(G)$. \square

Recall Theorem 2:

Theorem 2 (Bresar et. al. [1]) *For every tree T of order at least 2, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$.*

Proof. We proceed by induction on the order $n \geq 2$ of a tree T . If $n = 2$, then $T = K_2$ and $\gamma_{\text{pr}}(T) = 2 = 2\rho_3(T)$. This establishes the base case. Assume then that $n \geq 3$ and that all nontrivial trees T' of order less than n satisfy $\gamma_{\text{pr}}(T') = 2\rho_3(T')$. Let T be a tree of order n . If T is star or a double star, then $\gamma_{\text{pr}}(T) = 2 = 2\rho_3(T)$. Hence we may assume that $\text{diam}(T) \geq 4$.

In the proof we shall frequently prune the tree T to a tree T' and then establish that $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T') + 2k$ and $\rho_3(T) \geq \rho_3(T') + k$ for some integer $k \geq 0$. Since $\gamma_{\text{pr}}(T) \geq 2\rho_3(T)$ and $\gamma_{\text{pr}}(T') = 2\rho_3(T')$, it then follows that $\gamma_{\text{pr}}(T') + 2k \geq \gamma_{\text{pr}}(T) \geq 2\rho_3(T) \geq 2(\rho_3(T') + k) = \gamma_{\text{pr}}(T') + 2k$, whence $\gamma_{\text{pr}}(T) = 2\rho_3(T)$, as desired.

Suppose T has a strong support vertex v . Let u be a leaf neighbor of v , and let $T' = T - u$. Any $\gamma_{\text{pr}}(T')$ -set contains the support vertex v and is therefore a PDS of T , and so $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T')$. Any $\rho_3(T')$ -set is also a 3-packing in T , and so $\rho_3(T) \geq \rho_3(T')$. Thus, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$. Hence we may assume that T has no strong support vertex.

Let T be rooted at a leaf r of a longest path P . Let P be a r - u path, and let v be the neighbor of u . Then, u is a leaf of T and, since T has no strong support vertex, $\deg_T v = 2$. Let w denote the parent of v on this path and x the parent of w .

Suppose $\deg_T w \geq 3$ and w is a support vertex. Let v' be the leaf-neighbor of w , and let $T' = T - v'$. Then there exists a $\gamma_{\text{pr}}(T')$ -set that contains w (if w is not in some $\gamma_{\text{pr}}(T')$ -set,

then u and v are paired in such a $\gamma_{\text{pr}}(T')$ -set and we can simply replace u with the vertex w thereby pairing v and w in the new $\gamma_{\text{pr}}(T')$ -set). Such a PDS is also a PDS of T , and so $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T')$. Clearly, $\rho_3(T) \geq \rho_3(T')$. Thus, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$.

Suppose $\deg_T w \geq 3$ and w is not a support vertex. Then, each child of w is a support vertex of degree 2. Let v' be a child of w different from v , and let u' be the leaf-neighbor of v' . Let $T' = T - u' - v'$. Any $\gamma_{\text{pr}}(T')$ -set can be extended to a PDS of T by adding to it the vertices u' and v' (with u' and v' paired), and so $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T') + 2$. Let S' be a $\rho_3(T')$ -set that contains as many leaves as possible. Then, $S' \cap N[w] = \emptyset$ (for example, if S' contains a vertex from the set $\{v, w, x\}$, then we can simply replace such a vertex with the vertex u). Hence, S' can be extended to a 3-packing of T by adding to it the leaf u' , and so $\rho_3(T) \geq |S'| + 1 = \rho_3(T') + 1$. Thus, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$. Hence we may assume that $\deg_T w = 2$ for otherwise $\gamma_{\text{pr}}(T) = 2\rho_3(T)$, as desired.

Suppose $\deg_T x = 2$. Let $T' = T - \{u, v, w, x\}$. Any $\text{pr}(T')$ -set can be extended to a PDS of T by adding to it the vertices v and w (with v and w paired), and so $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T') + 2$. Any $\rho_3(T')$ -set can be extended to a 3-packing of T by adding to it the vertex u , and so $\rho_3(T) \geq \rho_3(T') + 1$. Thus, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$. Hence we may assume $\deg_T x \geq 3$.

Let $T' = T - \{u, v, w\}$. Every $\gamma_{\text{pr}}(T')$ -set can be extended to a PDS of T by adding to it the vertices v and w (with v and w paired), and so $\gamma_{\text{pr}}(T) \leq \gamma_{\text{pr}}(T') + 2$. Every $\rho_3(T')$ -set that does not contain x (if x belongs to a some $\rho_3(T')$ -set, then we can simply replace x with a child of x in T') can be extended to a 3-packing of T by adding to it the vertex u , and so $\rho_3(T) \geq \rho_3(T') + 1$. Thus, $\gamma_{\text{pr}}(T) = 2\rho_3(T)$. \square